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The Hamiltonian description of a second-order ODE

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Abstract
Making use of the formalism of differential forms, it is shown that any second-order ODE, or any system of two first-order ODEs, with differentiable coefficients, can be expressed in the form of the Hamilton equations with the Hamiltonian function being a differentiable function that can be chosen arbitrarily. It is also shown that any nontrivial local one-parameter group of symmetries of a second-order ODE, or a system of two first-order ODEs, is associated with a first integral.

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1. Introduction

The advantages of expressing the equations of motion of a mechanical system in Lagrangian or Hamiltonian form are widely known. One of them comes from the relationship between symmetries and constants of motion. When a system of ODEs is derivable from a Lagrangian one can easily find a first integral coming from each Noether symmetry (see, e.g. [1] for details) and each point symmetry can be used to reduce the order of the equations by 2 (see, e.g. [2]; here we shall focus on the Hamiltonian formalism, which allows us to consider more general symmetries). What is not so widely recognized is the fact that any second-order ODE with differentiable coefficients can be written in Hamiltonian form in infinitely many ways [3].

The aim of this paper is to give a simple, constructive proof of the fact that a system of two first-order ODEs with differentiable coefficients can always be written in the form of the Hamilton equations, where the Hamiltonian is an arbitrary differentiable function. We also show that with the infinitesimal generator of any nontrivial symmetry of the system of equations there is associated a first integral, even if the corresponding transformations are not canonical, or if the Hamiltonian is not invariant under the transformations.

In section 2, some elementary facts about the description of a system of two first-order ODEs in terms of vector fields and differential forms are briefly presented. In section 3,

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making use of differential forms, we show that any system of two first-order ODEs, or any second-order ODE, with differentiable coefficients can be written as the Hamilton equations with a Hamiltonian that is an arbitrary differentiable function. In section 4, we show that any vector field that represents a nontrivial symmetry of a system of two first-order ODEs yields a first integral, and conversely. Finally, in section 5 we present some examples, most of which correspond to second-order ODEs considered in books on symmetry methods for the solution of differential equations. This is done in order to facilitate the comparison of the results obtained by means of the approach presented in this paper with those obtained making use of other methods.

2. Preliminaries

In what follows we shall consider a real differentiable manifold P, of dimension 2 and x, y will denote a set of local coordinates for P. A pair of first-order ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t), \tag{1}$$

where f and g are smooth real-valued functions defined on $P \times \mathbb{R}$, is associated with, or described by, any of the following three objects: the vector field on $P \times \mathbb{R}$

$$\mathbf{A} \equiv \frac{\partial}{\partial t} + f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}, \tag{2}$$

whose integral curves, parameterized by t, are determined by the system of equations (1); the set of 1-forms on $P \times \mathbb{R}$

$$\alpha^{(1)} \equiv \mathrm{d}x - f(x, y, t)\,\mathrm{d}t, \qquad \alpha^{(2)} \equiv \mathrm{d}y - g(x, y, t)\,\mathrm{d}t, \tag{3}$$

which defines an integrable distribution of dimension 1; and the 2-form on $P \times \mathbb{R}$

$$\alpha^{(1)} \wedge \alpha^{(2)} = \mathrm{d}x \wedge \mathrm{d}y + (f\,\mathrm{d}y - g\,\mathrm{d}x) \wedge \mathrm{d}t. \tag{4}$$

The vector field $\mathbf{A}$ is the only vector field on $P \times \mathbb{R}$ whose contractions with $\alpha^{(1)}$ and $\alpha^{(2)}$ vanish,

$$\mathbf{A} \lrcorner \alpha^{(1)} = 0 = \mathbf{A} \lrcorner \alpha^{(2)}, \tag{5}$$

and satisfies the normalization condition

$$\mathbf{A}t = 1. \tag{6}$$

Equivalently, $\mathbf{A}$ is the only vector field satisfying equation (6), whose contraction with the 2-form (4) vanishes (that is, $\mathbf{A} \lrcorner (\alpha^{(1)} \wedge \alpha^{(2)}) = 0$).

As we shall show below, apart from these three descriptions, there is always an infinite number of symplectic structures and of Hamiltonian functions such that the system (1) can be written in the Hamiltonian form

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \tag{7}$$

where q, p, t is a coordinate system on $P \times \mathbb{R}$. In some cases, making use of the Legendre transformation, one can also find an infinite number of Lagrangian functions such that the system (1) is equivalent to the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{8}$$

A second-order ODE of the form $\mathrm{d}^2x/\mathrm{d}t^2 = F(x, \mathrm{d}x/\mathrm{d}t, t)$ can be expressed in the form (1), with $f(x, y, t) = y$ and $g(x, y, t) = F(x, y, t)$. Therefore, the results derived in what follows for a system of equations (1), will also apply to any second-order equation given in the form mentioned above.

Whereas the vector field $\mathbf{A}$ is uniquely defined, instead of the 1-forms $\alpha^{(1)}$ and $\alpha^{(2)}$, we can make use of any pair of 1-forms that can be expressed in the form $M^{(1)}_{(1)}\alpha^{(1)} + M^{(1)}_{(2)}\alpha^{(2)}$, $M^{(2)}_{(1)}\alpha^{(1)} + M^{(2)}_{(2)}\alpha^{(2)}$, where $M^{(i)}_{(j)}$ are functions such that $\det\left(M^{(i)}_{(j)}\right) \neq 0$. (The integrability of the distribution defined by$\alpha^{(1)}$ and $\alpha^{(2)}$ means that there exist functions $M^{(i)}_{(j)}$ such that

$$M^{(1)}_{(1)}\alpha^{(1)} + M^{(1)}_{(2)}\alpha^{(2)} = \mathrm{d}\varphi^{(1)}, \qquad M^{(2)}_{(1)}\alpha^{(1)} + M^{(2)}_{(2)}\alpha^{(2)} = \mathrm{d}\varphi^{(2)}, \tag{9}$$

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are real-valued functions. The solution of the system (1) is given by $\varphi^{(1)} = \text{const.}$ and $\varphi^{(2)} = \text{const.}$ The functions $\varphi^{(1)}$ and $\varphi^{(2)}$ are two functionally independent solutions of the linear PDE $\mathbf{A}\varphi^{(i)} = 0$.) Correspondingly, the 2-form $\alpha^{(1)} \wedge \alpha^{(2)}$ can be substituted by any nonzero multiple of it. Furthermore, any nonzero 2-form, Ω, on $P \times \mathbb{R}$ such that $\mathbf{A}\rfloor\Omega = 0$, must be a multiple of $\alpha^{(1)} \wedge \alpha^{(2)}$.

Any given system of equations (1) possesses an infinite number of (possibly local) one-parameter groups of diffeomorphisms that map each solution of equations (1) into a solution of these equations. These symmetry groups are conveniently studied by considering their infinitesimal generators. The vector field $\mathbf{X}$ on $P \times \mathbb{R}$ is the infinitesimal generator of a symmetry of the system (1) if and only if there exists a function λ such that

$$[\mathbf{X}, \mathbf{A}] = \lambda\mathbf{A} \tag{10}$$

(see, e.g. [1]). Any vector field proportional to $\mathbf{A}$ generates trivial symmetries in the sense that the corresponding transformations map a solution of equations (1) into itself.

One can show that equation (10) is equivalent to the existence of real-valued functions $N^{(i)}_{(j)}$ on $P \times \mathbb{R}$ such that $\pounds_{\mathbf{X}}\alpha^{(i)} = N^{(i)}_{(1)}\alpha^{(1)} + N^{(i)}_{(2)}\alpha^{(2)}$, where $\pounds_{\mathbf{X}}$ denotes the Lie derivative with respect to $\mathbf{X}$ and to the existence of a function ν such that

$$\pounds_{\mathbf{X}}(\alpha^{(1)} \wedge \alpha^{(2)}) = \nu\alpha^{(1)} \wedge \alpha^{(2)}. \tag{11}$$

3. The Hamiltonian form

In terms of the coordinates q, p, t, appearing in equations (7), the vector field $\mathbf{A}$ has the form

$$\mathbf{A} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

and therefore the 2-form (4) must be proportional to the *closed* 2-form

$$\Omega \equiv \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}H \wedge \mathrm{d}t. \tag{12}$$

Hence, there must exist a nonzero function, σ, such that

$$\sigma^{-1}[\mathrm{d}x \wedge \mathrm{d}y + (f\,\mathrm{d}y - g\,\mathrm{d}x) \wedge \mathrm{d}t] = \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}H \wedge \mathrm{d}t. \tag{13}$$

In fact, the differential of the 2-form on the left-hand side of equation (13) must be equal to zero, which amounts to the condition

$$\frac{\partial\sigma}{\partial t} + f\frac{\partial\sigma}{\partial x} + g\frac{\partial\sigma}{\partial y} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\sigma. \tag{14}$$

The function σ is defined up to a multiplicative function constant along the integral curves of $\mathbf{A}$ (that is, given a particular solution, σ_p, of (14), any other solution of this equation is of the form $F(\varphi^{(1)}, \varphi^{(2)})\sigma_p$, where F is an arbitrary nonvanishing function of two variables, and $\varphi^{(1)}, \varphi^{(2)}$ are two functionally independent solutions of $\mathbf{A}\varphi^{(i)} = 0$, as above).

Equation (14) implies that there exists a differentiable real-valued function H defined on $P \times \mathbb{R}$ such that $\sigma^{-1}(f\,\mathrm{d}y - g\,\mathrm{d}x) = \mathrm{d}H +$ terms proportional to $\mathrm{d}t$, if and only if $\partial\sigma/\partial t = 0$. Hence, in order to satisfy equation (13) in all cases, we introduce two auxiliary functions, ψ and ϕ, such that

$$\frac{\partial}{\partial x}[(f - \psi)\sigma^{-1}] + \frac{\partial}{\partial y}[(g - \phi)\sigma^{-1}] = 0. \tag{15}$$

This condition guarantees the existence of a function H such that

$$\sigma^{-1}[(f - \psi)\,\mathrm{d}y - (g - \phi)\,\mathrm{d}x] = \mathrm{d}H + \text{terms proportional to } \mathrm{d}t \tag{16}$$

and, therefore, the left-hand side of equation (13) can be expressed as

$$\begin{aligned}\sigma^{-1}[\mathrm{d}x \wedge \mathrm{d}y + (f\,\mathrm{d}y - g\,\mathrm{d}x) \wedge \mathrm{d}t] &= \sigma^{-1}\{(\mathrm{d}x - \psi\,\mathrm{d}t) \wedge (\mathrm{d}y - \phi\,\mathrm{d}t) \\ &\quad + [(f - \psi)\,\mathrm{d}y - (g - \phi)\,\mathrm{d}x)] \wedge \mathrm{d}t\} \\ &= \sigma^{-1}(\mathrm{d}x - \psi\,\mathrm{d}t) \wedge (\mathrm{d}y - \phi\,\mathrm{d}t) + \mathrm{d}H \wedge \mathrm{d}t,\end{aligned}$$

which, compared with equation (13), implies that the coordinates q and p must be obtained from

$$\sigma^{-1}(\mathrm{d}x - \psi\,\mathrm{d}t) \wedge (\mathrm{d}y - \phi\,\mathrm{d}t) = \mathrm{d}q \wedge \mathrm{d}p. \tag{17}$$

(It can be readily verified that the 2-form on the left-hand side of the last equation is indeed closed, as a consequence of equations (14) and (15).)

Summarizing, we have shown that the system of equations (1) can always be expressed in the Hamiltonian form (7) with the coordinates q and p defined by equation (17), where σ is a solution of (14) and ψ, ϕ are chosen in such a way that equation (15) is satisfied. The Hamiltonian, H, is then defined, up to an additive function of t only, by equation (16). (See the examples in section 5.)

As pointed out already, the function σ is defined up to a multiplicative function of two variables. Then, for a given choice of σ, the functions ψ and ϕ are defined up to the transformation

$$\psi \mapsto \psi - \sigma\frac{\partial\Lambda}{\partial y}, \qquad \phi \mapsto \phi + \sigma\frac{\partial\Lambda}{\partial x}, \tag{18}$$

where Λ is an arbitrary function of x, y, t. Equation (16) shows that the transformation (18) is accompanied by the transformation

$$H \mapsto H + \Lambda. \tag{19}$$

Since Λ is an arbitrary function, the Hamiltonian can be transformed into any function we like (including the function identically zero). The coordinates q and p are defined by equation (17) up to a time-independent canonical transformation.

Remark 1. Choosing $\psi = f$ and $\phi = g$, equation (15) is trivially satisfied and equation (16) shows that we can take $H = 0$. Hence, the canonical coordinates q and p obtained from

$$\sigma^{-1}(\mathrm{d}x - f\,\mathrm{d}t) \wedge (\mathrm{d}y - g\,\mathrm{d}t) = \mathrm{d}q \wedge \mathrm{d}p \tag{20}$$

(see equation (17)) are constant. (An example is given in section 5, see equation (31).) This is equivalent to what one would get by finding a complete solution of the Hamilton–Jacobi equation.

4. Symmetries and first integrals

In this section, we shall show that *any* vector field, $\mathbf{X}$, on $P \times \mathbb{R}$ that generates a nontrivial symmetry of the system of equations (1) is associated with a first integral, even if the corresponding transformations are not canonical or the Hamiltonian is not invariant under the transformations generated by $\mathbf{X}$ (see the examples below).

Proposition. *If the vector field* $\mathbf{X}$ *is a symmetry of the system of equations (1), then the 1-form* $\mathbf{X}\rfloor\Omega$ *is integrable and, if* $\mathbf{X}\rfloor\Omega \neq 0$*, then* $\mathbf{X}\rfloor\Omega$ *is proportional to the differential of a first integral of the equations (1) (that is,* $\mathbf{X}\rfloor\Omega = \mu\, \mathrm{d}\chi$*, with* $\mathbf{A}\chi = 0$*). Conversely, if* χ *is a first integral (that is,* $\mathbf{A}\chi = 0$*), then there exists a vector field* $\mathbf{X}$ *on* $P \times \mathbb{R}$ *that is a symmetry of the system of equations (1) and satisfies* $\mathbf{X}\rfloor\Omega = \mathrm{d}\chi$.

Proof. Making use of the identity $\pounds_{\mathbf{X}}\Omega = \mathbf{X}\rfloor\, \mathrm{d}\Omega + \mathrm{d}(\mathbf{X}\rfloor\Omega)$ and of the fact that Ω is closed, we have $\pounds_{\mathbf{X}}\Omega = \mathrm{d}(\mathbf{X}\rfloor\Omega) = \nu\Omega$. Hence, using the fact that $\Omega = \sigma^{-1}\alpha^{(1)} \wedge \alpha^{(2)}$, and that the contraction is an antiderivation we have

$$\begin{aligned}(\mathbf{X}\rfloor\Omega) \wedge \mathrm{d}(\mathbf{X}\rfloor\Omega) &= (\mathbf{X}\rfloor\Omega) \wedge (\nu\Omega)\\ &= \nu\sigma^{-2}[\mathbf{X}\rfloor(\alpha^{(1)} \wedge \alpha^{(2)})] \wedge \alpha^{(1)} \wedge \alpha^{(2)}\\ &= \nu\sigma^{-2}[(\mathbf{X}\rfloor\alpha^{(1)})\alpha^{(2)} - (\mathbf{X}\rfloor\alpha^{(2)})\alpha^{(1)}] \wedge \alpha^{(1)} \wedge \alpha^{(2)}\\ &= 0\end{aligned}$$

(see equations (5)) thus showing that the 1-form $\mathbf{X}\rfloor\Omega$ is integrable; that is, there exist functions μ and χ such that $\mathbf{X}\rfloor\Omega = \mu\, \mathrm{d}\chi$. Then, using the skew-symmetry of Ω and the fact that the contraction of $\mathbf{A}$ with Ω vanishes, we see that

$$\begin{aligned}\mu\mathbf{A}\chi &= \mathbf{A}\rfloor(\mu\, \mathrm{d}\chi)\\ &= \mathbf{A}\rfloor(\mathbf{X}\rfloor\Omega)\\ &= -\mathbf{X}\rfloor(\mathbf{A}\rfloor\Omega)\\ &= 0.\end{aligned}$$

Conversely, if χ is a first integral, then there exist functions $\beta_{(1)}$ and $\beta_{(2)}$ such that

$$\mathrm{d}\chi = \beta_{(1)}\alpha^{(1)} + \beta_{(2)}\alpha^{(2)}$$

(see equations (5)). Let $\mathbf{X}$ be a vector field such that

$$\mathbf{X}\rfloor\alpha^{(1)} = \sigma\beta_{(2)}, \qquad \mathbf{X}\rfloor\alpha^{(2)} = -\sigma\beta_{(1)} \tag{21}$$

(which defines $\mathbf{X}$ up to an additive multiple of $\mathbf{A}$). Then,

$$\begin{aligned}\mathbf{X}\rfloor\Omega &= \sigma^{-1}\mathbf{X}\rfloor(\alpha^{(1)} \wedge \alpha^{(2)})\\ &= \sigma^{-1}(\mathbf{X}\rfloor\alpha^{(1)})\alpha^{(2)} - \sigma^{-1}(\mathbf{X}\rfloor\alpha^{(2)})\alpha^{(1)}\\ &= \mathrm{d}\chi\end{aligned}$$

and, therefore, $\pounds_{\mathbf{X}}\Omega = \mathbf{X}\rfloor\, \mathrm{d}\Omega + \mathrm{d}(\mathbf{X}\rfloor\Omega) = 0$, thus showing that $\mathbf{X}$ is the infinitesimal generator of a symmetry of the system of equations (1). □

Remark 2. If the Lie derivative of Ω with respect to $\mathbf{X}$ is equal to zero (that is, $\nu = 0$), then the contraction $\mathbf{X}\rfloor\Omega$ is (locally) exact, $\mathbf{X}\rfloor\Omega = \mathrm{d}\chi$, with $\mathbf{A}\chi = 0$.

Remark 3. Despite the fact that there is an infinite number of Hamiltonians for a given system of equations (1), the results of this section show that the first integral associated with a symmetry is defined in an essentially unique way.

5. Examples

In this final section we give three examples where we show how to apply the algorithms given in the last two sections.

5.1. Forced damped harmonic oscillator

A first example is given by the linear second-order ODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = \eta(t), \tag{22}$$

where γ and ω are real constants, and η is some function of one variable. This equation is equivalent to the system of equations (1) taking

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y - \omega^2 x + \eta(t),$$

that is, $f(x, y, t) = y$, $g(x, y, t) = -\gamma y - \omega^2 x + \eta(t)$. According to equation (14), σ has to satisfy the equation

$$\frac{\partial\sigma}{\partial t} + f\frac{\partial\sigma}{\partial x} + g\frac{\partial\sigma}{\partial y} = -\gamma\sigma.$$

We can take $\sigma = \exp(-\gamma t)$ (any other choice would involve the knowledge of a first integral of equation (22)) and equation (15) reduces to

$$\frac{\partial\psi}{\partial x} + \frac{\partial}{\partial y}(\gamma y + \phi) = 0.$$

Choosing $\phi = 0$ and $\psi = -\gamma x$, equation (17) yields $\mathrm{d}(x\,\mathrm{e}^{\gamma t}) \wedge \mathrm{d}y = \mathrm{d}q \wedge \mathrm{d}p$; therefore, we can take $q = x\mathrm{e}^{\gamma t}$ and $p = y$. Then, making use of equation (16) we find that the Hamiltonian is given by

$$\begin{aligned} H &= \mathrm{e}^{\gamma t}\left(\frac{y^2}{2} + \gamma xy + \frac{\omega^2 x^2}{2} - x\eta(t)\right) \\ &= \mathrm{e}^{\gamma t}\frac{p^2}{2} + \gamma pq + \mathrm{e}^{-\gamma t}\frac{\omega^2 q^2}{2} - q\eta(t). \end{aligned} \tag{23}$$

The corresponding Lagrangian, expressed in terms of the original variables, is

$$L = \mathrm{e}^{\gamma t}\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 + x\eta(t)\right).$$

Remark 4. In order to find a first integral associated with a symmetry of the system of equations (1) it is not necessary to find the 'integrating factor' σ.

For instance, when $\eta = 0$, equation (22) becomes homogeneous, in the sense that the multiplication of x by a constant factor yields another solution of this equation. A vector field corresponding to this symmetry is

$$\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{24}$$

According to the foregoing proposition, the 1-form

$$\mathbf{X}\rfloor(\alpha^{(1)} \wedge \alpha^{(2)}) = x\,\mathrm{d}y - y\,\mathrm{d}x + (\omega^2 x^2 + \gamma xy + y^2)\,\mathrm{d}t$$

must be proportional to the differential of a first integral. Following a standard procedure (see, e.g., [4]) one finds that

$$\mathbf{X}\rfloor(\alpha^{(1)} \wedge \alpha^{(2)}) = (\omega^2 x^2 + \gamma x y + y^2)\, \mathrm{d}\left(t + \frac{1}{r_1 - r_2} \ln\left|\frac{y - r_1 x}{y - r_2 x}\right|\right),$$

where r_1 and r_2 are the roots of the polynomial $r^2 + \gamma r + \omega^2 = 0$ (which we assume distinct).

One of the various consequences of having a set of equations expressed in the form of the Hamilton equations is that the volume in the phase space is preserved under the time evolution (Liouville's theorem). In the present example, such a behavior seems strange since, when $\eta = 0$ (damped harmonic oscillator) any initial condition evolves approaching the equilibrium point $x = 0$, $y = 0$. There is no contradiction, however, owing to the factor $\mathrm{e}^{\gamma t}$ contained in the coordinate transformation $q = x\,\mathrm{e}^{\gamma t}$, $p = y$, which compensates the shrinking of the phase space area element $\mathrm{d}x \wedge \mathrm{d}y$.

5.2. A linear equation

A second example is provided by the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2y - x + \frac{y - x}{t} \tag{25}$$

[1, example 7.5.3], which is of the form (1), with $f(x, y, t) = y$, $g(x, y, t) = 2y - x + (y - x)/t$; hence, from equation (14), we find that σ can be taken as $\sigma = t\mathrm{e}^{2t}$. Then equation (15) is trivially satisfied if we choose $\psi = 0$, $\phi = 2y + y/t$, and equation (16) shows that the Hamiltonian is then given by

$$H = \mathrm{e}^{-2t}\left(\frac{y^2}{2t} + \frac{x^2}{2t} + \frac{x^2}{2t^2}\right). \tag{26}$$

A set of canonical coordinates is obtained from equation (17), which, in the present case, reads

$$t^{-1}\,\mathrm{e}^{-2t}\,\mathrm{d}x \wedge \left[\mathrm{d}y - \left(2y + \frac{y}{t}\right)\mathrm{d}t\right] = \mathrm{d}q \wedge \mathrm{d}p.$$

Hence, modulo a canonical transformation, $q = x$, and $p = yt^{-1}\mathrm{e}^{-2t}$. In terms of these variables, the Hamiltonian is

$$H = \frac{\mathrm{e}^{2t} t p^2}{2} + \frac{\mathrm{e}^{-2t} q^2}{2t} + \frac{\mathrm{e}^{-2t} q^2}{2t^2} \tag{27}$$

and the Lagrangian

$$L = \mathrm{e}^{-2t}\left(\frac{\dot{q}^2}{2t} - \frac{q^2}{2t} - \frac{q^2}{2t^2}\right). \tag{28}$$

According to equations (12) and (13), in the present example the closed 2-form Ω, in terms of the original variables, is given by

$$\Omega = t^{-1}\mathrm{e}^{-2t}\left\{\mathrm{d}x \wedge \mathrm{d}y + \left[y\,\mathrm{d}y - \left(2y - x + \frac{y - x}{t}\right)\mathrm{d}x\right] \wedge \mathrm{d}t\right\}. \tag{29}$$

The vector field

$$\mathbf{X} = \mathrm{e}^t \frac{\partial}{\partial x} + \mathrm{e}^t \frac{\partial}{\partial y} \tag{30}$$

corresponds to a symmetry of the system of equations (25) (in fact, $[\mathbf{X}, \mathbf{A}] = 0$). A straightforward computation, making use of expression (29), shows that $\pounds_{\mathbf{X}}\Omega = 0$ and

$$\begin{aligned}\mathbf{X}\rfloor\Omega &= t^{-1}\,\mathrm{e}^{-t}\left[\mathrm{d}y - \mathrm{d}x + \left(x - y + \frac{x - y}{t}\right)\mathrm{d}t\right] \\ &= \mathrm{d}[(y - x)t^{-1}\,\mathrm{e}^{-t}].\end{aligned}$$

As one can readily verify, making use of equations (25), the function $\varphi^{(1)} \equiv (y-x)t^{-1}\,\mathrm{e}^{-t}$ is indeed a first integral. (Compare with the standard procedures employed to solve second-order ODEs using symmetries given, e.g., in [1, 2], where a change of coordinates is required or the existence of two or more symmetries.) In this case it is not difficult to find a function $\varphi^{(2)}$ such that $\Omega = \mathrm{d}\varphi^{(1)} \wedge \mathrm{d}\varphi^{(2)}$, with Ω given by equation (29), e.g.,

$$\varphi^{(2)} = \mathrm{e}^{-t}\left(-x - \tfrac{1}{2}tx + \tfrac{1}{2}ty\right). \tag{31}$$

By comparing with equation (12), one concludes that $\varphi^{(1)}$ and $\varphi^{(2)}$ are canonically conjugate variables with a Hamiltonian equal to zero. Hence, $\varphi^{(1)}$ and $\varphi^{(2)}$ are two functionally independent first integrals. (It may be noticed that, even though $\pounds_{\mathbf{X}}\Omega = 0$, the Hamiltonian (26) is not invariant under the transformations generated by (30), i.e., $\mathbf{X}H \neq 0$.)

In order to give an example related to the second part of the proposition of section 4, we can find a symmetry associated with the conservation of $\varphi^{(2)} = \mathrm{e}^{-t}\big(-x - \frac{1}{2}tx + \frac{1}{2}ty\big)$ (see equation (31)). In fact,

$$\mathrm{d}\varphi^{(2)} = -\left(1 + \tfrac{1}{2}t\right)\mathrm{e}^{-t}\alpha^{(1)} + \tfrac{1}{2}\,\mathrm{e}^{-t}\alpha^{(2)}.$$

Hence, conditions (21) take the explicit form

$$\mathbf{X}\rfloor(\mathrm{d}x - y\,\mathrm{d}t) = \tfrac{1}{2}t^2\,\mathrm{e}^t, \qquad \mathbf{X}\rfloor\{\mathrm{d}y - [2y - x + (y-x)/t]\,\mathrm{d}t\} = \left(t + \tfrac{1}{2}t^2\right)\mathrm{e}^t$$

and, therefore, $\mathbf{X}$ can be chosen as

$$\mathbf{X} = \tfrac{1}{2}t^2\,\mathrm{e}^t\frac{\partial}{\partial x} + \left(t + \tfrac{1}{2}t^2\right)\mathrm{e}^t\frac{\partial}{\partial y}. \tag{32}$$

It can be readily verified that this last vector field is indeed a symmetry of the system of equations (25). (Moreover, the vector field (32) corresponds to a Lie point symmetry of the Lagrangian (28).)

5.3. *The Emden–Fowler equation*

Another illustrative example is provided by the Emden–Fowler equation

$$\ddot{x} + \frac{2\dot{x}}{t} + x^n = 0 \tag{33}$$

[5, section 3.2], where n is a constant. This equation, which appears in the study of a self-gravitating gas, can be expressed as a system of the form (1) with

$$f(x,y,t) = y, \qquad g(x,y,t) = -\frac{2y}{t} - x^n. \tag{34}$$

Then, from equation (14) one readily finds that σ can be chosen as

$$\sigma = t^{-2}$$

and equation (15) is satisfied if we take

$$\psi = 0, \qquad \phi = -\frac{2y}{t}.$$

Substituting these expressions into equation (16) we obtain

$$t^2(y\,\mathrm{d}y + x^n\,\mathrm{d}x) = \mathrm{d}\left(\frac{t^2y^2}{2} + \frac{t^2x^{n+1}}{n+1}\right) + \text{terms proportional to } \mathrm{d}t.$$

Hence, we can take

$$H = \frac{t^2y^2}{2} + \frac{t^2x^{n+1}}{n+1}. \tag{35}$$

On the other hand, equation (17) gives

$$\mathrm{d}q \wedge \mathrm{d}p = t^2\,\mathrm{d}x \wedge \left(\mathrm{d}y + \frac{2y}{t}\,\mathrm{d}t\right) = \mathrm{d}x \wedge \mathrm{d}(t^2 y)$$

and, therefore, we can choose $q = x$, $p = t^2 y$. In terms of these canonical coordinates, the Hamiltonian (35) takes the form

$$H = \frac{p^2}{2t^2} + \frac{t^2 q^{n+1}}{n+1}.$$

By inspection, one finds that equation (33) has scaling symmetry, which is generated by the vector field

$$\mathbf{X} = t\frac{\partial}{\partial t} - \frac{2}{n-1}x\frac{\partial}{\partial x} - \frac{n+1}{n-1}y\frac{\partial}{\partial y}. \tag{36}$$

In fact, $\pounds_{\mathbf{X}}\Omega = \frac{n-5}{n-1}\Omega$. Thus, according to the preceding results, $\mathbf{X}\rfloor\Omega$ is proportional to the differential of a first integral, and, in the special case where $n = 5$ (which is the only one considered in some detail in [5]), $\pounds_{\mathbf{X}}\Omega = 0$, which implies that $\mathbf{X}\rfloor\Omega$ is exact. In fact, one readily finds that, when $n = 5$,

$$\mathbf{X}\rfloor\Omega = -\tfrac{1}{6}\,\mathrm{d}(3t^3y^2 + 3t^2xy + t^3x^6).$$

(The fact that $3t^3y^2 + 3t^2xy + t^3x^6$ is a first integral is obtained in [5] making use of Lie's reduction theorem.)

6. Final remark

The results of this paper suggest that in order to find a Hamiltonian formulation in the case of systems of two or more second-order ODEs, it may be convenient to employ the formalism of differential forms, following closely the derivation presented here.

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